Relevance of Chaos and Strange Attractors in the Samuelson-Hicks Oscillator

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Abstract: In this paper, we look for the relevance of chaos in the well-known Hicks-Samuelson's oscillator model investigating the endogenous fluctuations of the national income between two limits: full employment income and under-employment income. We compute the Lyapunov exponent, via Monte-Carlo simulations, to detect chaos in the evolution of the income between both limits. In the case of positive Lyapunov exponent and large values of the parameter (i.e. marginal propensity to consume and technical coefficient for capital), the evolution of income is seen to be chaotic. The model also may contain a quasi-periodic attractor that can be chaotic or not.

Keywords: chaos; attractor; oscillator; post-Keynesian; business fluctuations.

JEL Classification: C62; E12; E20; E32.

1. Introduction

The original linear model of accelerator-multiplier developed by P.A. Samuelson (1939) relies on a multiplier mechanism, which is based on a simple Keynesian consumption with a lag, and investment, depending on the variation in consumption (determined by the level of economic activity), which involves the accelerator mechanism. The combination of these two mechanisms gives rise to Samuelson's oscillator.

In his paper, Samuelson explains how multiplier and acceleration generate business cycles and fluctuations in national income. Before this paper, there are only marginal references regarding the role of both these principles in the theory of economic fluctuations. It was Harrod (1936, p. 33) who incorporated the interaction between them in the theory of trade cycles in an article that he published in 1936 in parallel with the famous Keynes' *General Theory*.

Samuelson models this interaction by choosing several values of the marginal propensity to consume and the marginal coefficient of capital. According to certain values of these parameters, the evolution of national income exhibits oscillations. These oscillations may be damped, perfectly regular, or explosive. Although this model contains some valid elements regarding the explication of economic fluctuations, it is not able to produce lasting business cycles. Moreover, empirically observed values of its coefficients imply that the trajectory of income is unstable (Westerhoff, 2006).

Thus, improving Samuelson's model, J.R. Hicks (1950) adds some changes by indicating that in a stationary state, induced – as well as the total net investment – must be nil, and gross investment must be equal to depreciation. Furthermore, he adds a floor (the under-employment income) and a ceiling (the full employment income) in this model and formulates a piecewise linear framework that can produce bounded oscillations. He also adds a geometric growth model that can be coupled with the business cycles. Some authors find that "quasi-periodic attractors" can occur in the basic Hicks model and other authors investigate the mathematical properties of such a model (Gallegati, Gardini, Puu and Sushko, 2003).

In this paper, we show that, even though nonlinearity is a necessary (but not sufficient) condition for the occurrence of chaos in dynamical systems, the Samuelson-Hicks model displays chaos for plausible and widely used parameters values. Thus, we search the relevance of chaos characterised by quasi-periodic attractors by using Monte-Carlo simulation to estimate the Average Lyapunov Exponent whose value depends on the values of the marginal propensity to consume and the marginal coefficient of capital.

So, this paper contains the following sections. Section 2 provides a brief literature review of chaotic models. Moreover, it sheds light on the inclusion of such models in economics and particularly in the Samuelson-Hicks oscillator. Section 3 presents the original Samuelson-Hicks model and analyses the evolution of the national income between the floor and the ceiling. Section 4 exhibits the relevance of chaos by using Monte-Carlo simulation to estimate the Average Lyapunov Exponent, which is a useful tool for measuring chaos in the Samuelson-Hicks model. Section 5 indicates the possibility of the quasi-periodic attractor occurrence and makes a comparison between chaotic evolution and periodic, damped, or explosive oscillations of national income. Section 6 concludes.

2. Chaotic Model in the Samuelson-Hicks's Oscillator: A Literature Review

Non-linearity in economics is relevant, which is why several researchers have included chaos in their analysis, most frequently employing endogenous fluctuation models such as the Samuelson-Hicks oscillator.

2.1. Chaotic Models: An Overview

Chaos theory is primarily used in the meteorology fields (Lorenz, 1960; 1972). The main insight behind this concept is that even a simple deterministic system can sometimes produce unpredictable situations – notably when such a deterministic system has a sensitivity to initial conditions in the short run.

Traditionally, chaos theory is analysed by means of a logistic function used as a simple model of biological growth (Baumol and Benhabib, 1989) such as:

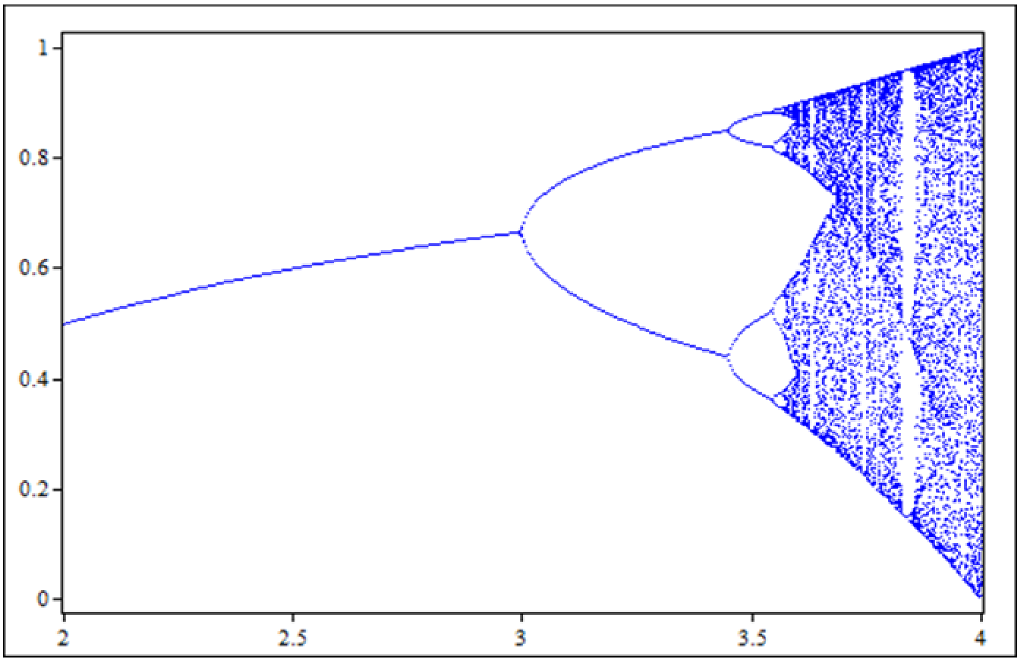
$$y_{t+1} = ay_t(1 - y_t) \text{ with } 0 < a < 4 \quad [1]$$

Figure 1, below, represents the evolution of the x variable (y-axis) as parameter a varies (x-axis).

This figure was shaped by simulating the evolution of the system over 10,000 iterations. It shows that if a system exhibits repeated periods of doubling then it will have an infinite number of bifurcations with a finite increase of that parameter (Feigenbaum, 1978).

For $0 \leq a \leq 1$, the stationary solution at the origin is stable and the system exhibits a cycle of period 1. If $1 \leq a \leq 3$, the stationary solution is stable but when $a = 3.1$, the system undergoes a bifurcation and presents a cycle of period 2. If $a > 3.1$ (and equal to around 3.5) the cycle of period 2 splits into a cycle of period 4. From $a = 3.57$ to 4, the system exhibits chaotic behaviour – except between the values ranging from about 3.82 to about 3.86, where a white window appears. This indicates that the system moves from chaos back into order, but it bifurcates again and returns to chaos at $a = 3.86$.

Figure 1. Evolution of the system as parameter *a* varies



In fact, as Faggini and Parziale (2014) point out, chaos means order without predictability and is not just an extension of standard economics.

2.2. Chaos in Economics and its Inclusion in Endogenous Fluctuations Models

Chaos constitutes a different way of seeing the economy, and the inclusion of this concept in economic analysis is not recent.

For example, Mandelbrot (1963) analyses the chaotic variation of speculative prices. Kesley (1988), using the overlapping generation model, asserts that economics models involve chaos. Baumol and Benhabib (1989) present nonlinear models as an example of chaos estimation.

More recently, Viad et al., (2010), taking an example of chaos in exchange rates, show that chaos theory is related to the notion of nonlinearity. Federici and Gandolfo (2014) propose various tests of chaotic behaviour in economics by also considering exchange rates. Other authors use chaos theory and the attractor approach to identify a chaotic dynamic in the evolution of GDP (Verne and Doueiry-Verne, 2019).

All these models are based on an econometric analysis taking into account the random factor via residuals of equations. However, chaos theory can also be used in endogenous fluctuations models that do not include the random factor.

The relevance of chaos in the endogenous fluctuations model was also discussed by Hommes (1995), Gallegati et al. (2003), Puu et al. (2005), and Piiroinen and Raghavendra (2019).

Hommes (1995, p. 436) analyses one of the simplest non-linear business cycle models introduced by Hicks by examining whether the path in Hicks's trade cycle model converges to a periodic time path every time. He extended the Hicks' model by considering lags in consumption and/or investment distributed over three time periods, and duly demonstrated the existence of quasi-periodic and strange attractors.

By referring to the dynamic of the multiplier-accelerator model, developed by Hommes (1995) and Piiroinen et al. (2019), we can show that the attractors exhibit periodic behaviour intercepted by a sudden burst of erratic behaviour – which is pertinent for understanding regime shifts that we encounter in real economies.

In a more detailed investigation of the dynamics of Hicks' model, Gallegati et al., (2003) analyse bifurcations to study the conditions under which the model produces periodic and quasi-periodic dynamics. Thus, using certain values of the parameters composing the linear model of Samuelson-Hicks, the authors (Gallegati et al., 2003, p. 514) analyse a two-dimensional bifurcation diagram and show that an attractive cycle of some period, or a quasi-periodic trajectory, can occur.

Puu et al. (2005) revisited the original issue of growth oscillations using the relative deviations approach. The authors suggest a reformulation of the Samuelson-Hicks oscillator model by asserting that 'it is not only arbitrary to assume the floor to grow at the same rate as the autonomous expenditures, but the change even goes in wrong direction'. In fact, 'the floor would rather be decreasing with capital accumulation' (Puu et al., 2005, pp. 333-334).

Piiroinen and Raghavendra (2019, p. 3) reconsider the Samuelson multiplier-accelerator model by introducing a discontinuity in the investment expenditure, as in the case of Hicksian extension. As a result, such a modification yields new dynamics in terms of periodic orbits and non-periodic attractors. In addition, their model can generate bounded dynamics without needing to employ a floor and ceiling in the region where the system is deemed to be unstable, as in the original model by Samuelson.

For our purpose, we assume the existence of a floor and ceiling a la the Samuelson-Hicks oscillator and indicate the relevance of chaos by simulating several values of parameters of variables forming the model.

3. The Samuelson-Hicks Model

As we saw, Hicks improves the Samuelson model by adding the rate of growth to the variables, as well as the ceiling and floor.

3.1. The Original Samuelson Model

Samuelson's original paper (1939, p. 76), contains four macroeconomic variables:
national income at time t; Y_t which is itself the sum of three components:
governmental expenditure, A_t;; consumption expenditure, C_t and private investment, I_t.

The first relationship between these four variables is an identity relation, as in the Keynesian tradition:

$$Y_t = C_t + I_t + A_t \quad [2]$$

In the Samuelson-Hicks model, investment is determined by the growth of income, through the principle of acceleration where investment is proportional to the rate of change in income:

$$I_t = k(Y_{t-1} - Y_{t-2}) \quad [3]$$

With *k*, the marginal coefficient of capital or the technical coefficient for capital e.g. the volume of capital needed to produce one unit of goods during one time period. Y_{t-1} and Y_{t-2}, are income of one and two periods back respectively.

The third relationship is about consumption expenditure function with the lagged income $Y_{t\text{-}1}$.

$$C_t = cY_{t\text{-}1} \qquad [4]$$

With *c*, the marginal propensity to consume, we can write the national income as:

$$Y_t = (\mathrm{c} + k)\, Y_{t\text{-}1} - kY_{t\text{-}2} + A_t \qquad [5]$$

From this equation, we can estimate the evolution of income according to the values of marginal propensity to consume and the technical coefficient for capital. For example, for large values of *c* and *k*, the national income records explosive oscillations while it presents perfectly periodic fluctuations when *k* = 1 and *c* = 0.5. If the *c* and *k* parameters take certain values, we obtain the inverted complex roots from equation [5] written in a polynomial form:

$$Y_t[1 - (c + k)L + kL^2] = A_t \qquad [6]$$

L is the lag operator where $L^k = Y_{t\text{-}k}$

Thus, in the case of oscillations, the determinant is $\Delta = (c + k)^2 - 4k < 0$ and $L = \frac{(c+k)}{2} \pm \frac{i\sqrt{\Delta}}{2}$

Setting $\frac{(c+k)}{2} = \alpha$ and $\frac{i\sqrt{\Delta}}{2} = \beta$, we calculate the modulus[1] $p = (\alpha^2 + \beta^2)^{0.5}$

In the Hicks model, the lower limit (the floor) is applied to induced investment while the upper limit (the ceiling) is applied to full employment (Gallegati, Gardini, Puu and Sushko, 2003, p. 508). In addition, Hicks models a growth process by introducing autonomous expenditures, which may be growing exponentially i.e. $A_t = A_0(1 + g)^t$ where g is a given growth rate and A_0 a positive constant. Therefore, the solution of the characteristic equation with complex roots is the product of exponential growth i.e. $Y_t = Y_0(1 + g)^t$.

By substituting the values of A_t and Y_t in [5] we define the stationary income and the two limits: the ceiling, e.g. the full employment income and the floor, the under-employment income where the induced investment is nil, and gross investment equals depreciation.

From equation [5] we can write:

$$Y_t = (c + k)\, Y_0(1 + g)^{t\text{-}1} - kY_0(1 + g)^{t\text{-}2} + A_0(1 + g)^t \qquad [7]$$

[1] If *p* < 1, the values of inverted roots are inside the unit circle of the complex plane and income oscillations are damped. The process is stationary, and the national income returns towards its long-run value.
If *p* > 1, the values of inverted roots are outside the unit circle of the complex plane and income oscillations are explosive.
If *p* = 1, national income oscillations exhibit perfectly sinusoidal fluctuations.

By substituting $Y_t = Y_0(1+g)^t$ in [7], we have:

$$Y_0(1+g)^t - (c+k)\,Y_0(1+g)^{t-1} + kY_0(1+g)^{t-2} = A_0(1+g)^t \qquad [8]$$

And:

$$Y_0(1+g)^{t-2}\,[\,(1+g)^2 - (c+k)\,(1+g) + k] = A_0(1+g)^t \qquad [9]$$

Finally, we obtain the stationary income or the equilibrium path:

$$Y_0 = \frac{A_0(1+g)^2}{[(1+g)^2-(c+k)(1+g)+k]} \qquad [10]$$

Equation [10] determines the equilibrium path around which the income Y_t may fluctuate.

In the Hicks model, we define the equilibrium growth path as:

$$Y_E = Y_0(1+g)^t$$

When the technical coefficient for capital $k > 1$, the national income leaves the equilibrium path and inevitably reaches the ceiling of full employment for a maximum of two periods. Then, during the recession, national income falls to the floor.

The equation of the full employment output path is as follows: $YM_t = YM_0(1+g)^t$.
YM_t is the full employment output at time t and YM_0 represents the full employment output at time 0 (e.g., the first term of the equation written in the form of geometric growth rate).

By substituting this term in the equation [5], we obtain:

$$YM_t = (c+k)\,YM_0(1+g)^{t-1} - kYM_0(1+g)^{t-2} + A_0(1+g)^t \qquad [11]$$

In fact:

$$(c+k)\,YM_0(1+g)^{t-1} - kYM_0(1+g)^{t-2} + A_0(1+g)^t < YM_0(1+g)^t \qquad [12]$$

Equation [12] is verified if:

$YM_0 > Y_0$ (computed in the relation [10]).

After two periods, a change in the trajectory of national income occurs. It is the beginning of the recession phase where the induced investment disappears due to the decline in production. Hence, $k = 0$, and the relation [5] is simplified:

$$Y_t = \mathrm{c}\,Y_{t-1} + A_t \qquad [13]$$

We define the under-employment output path (the floor) as $YL_t = YL_0(1+g)^t$. By using this term in the equation [13], we obtain:

$$YL_0(1+g)^t = c\, YL_0(1+g)^{t-1} + A_0(1+g)^t$$

By rearranging the terms, we have:

$$YL_0(1+g)^{t-1}(1 - g - c) = A_0(1+g)^t \qquad [14]$$

Finally, we compute the under-employment income as follows:

$$YL_0 = \frac{A_0(1+g)}{1-g-c} \qquad [15]$$

During the recession phase, Y_t falls to the under-employment level.

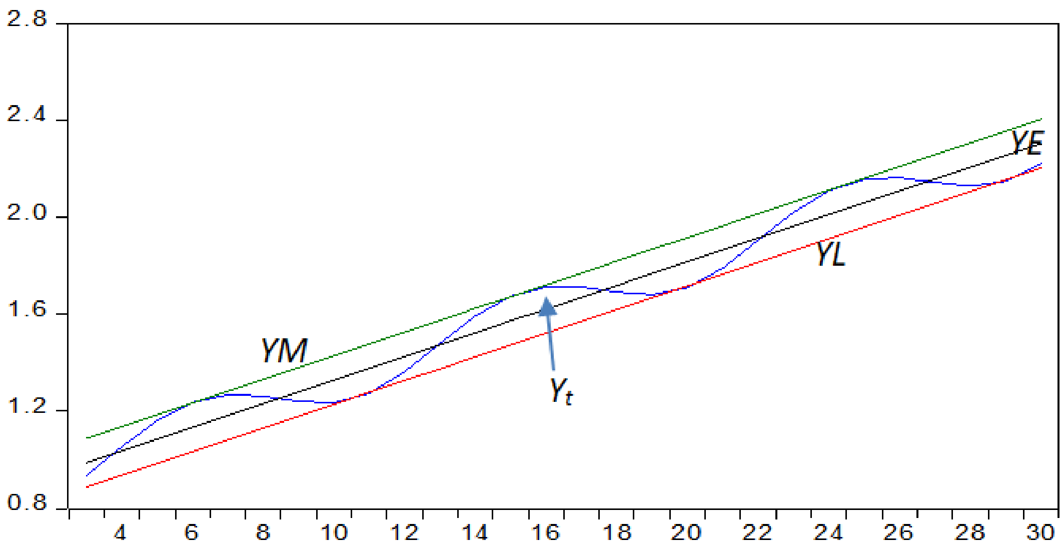
According to certain values of the marginal propensity to consume and the technical coefficient for capital, income displays several kinds of oscillations between both limits.

3.2. Evolution of Income Between Floor and Ceiling

In order to display the evolution of income between ceiling and floor, we assume several values for the technical coefficient for capital, k, and the marginal propensity to consume c. In addition, we take a period of 30 years and suppose that the economic growth rate is $g = 5\%$ per year.

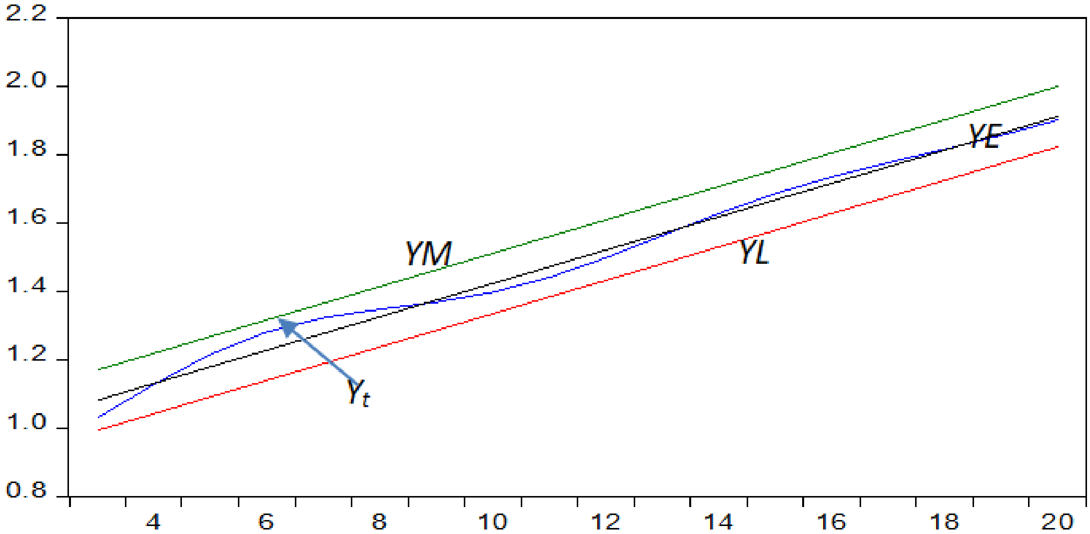
If we take the special case where $c = 0.5$ and $k = 1$, the evolution of income is seen to be perfectly sinusoidal between the two limits.

Figure 2. Sinusoidal Evolution of Income Between Ceiling and Floor



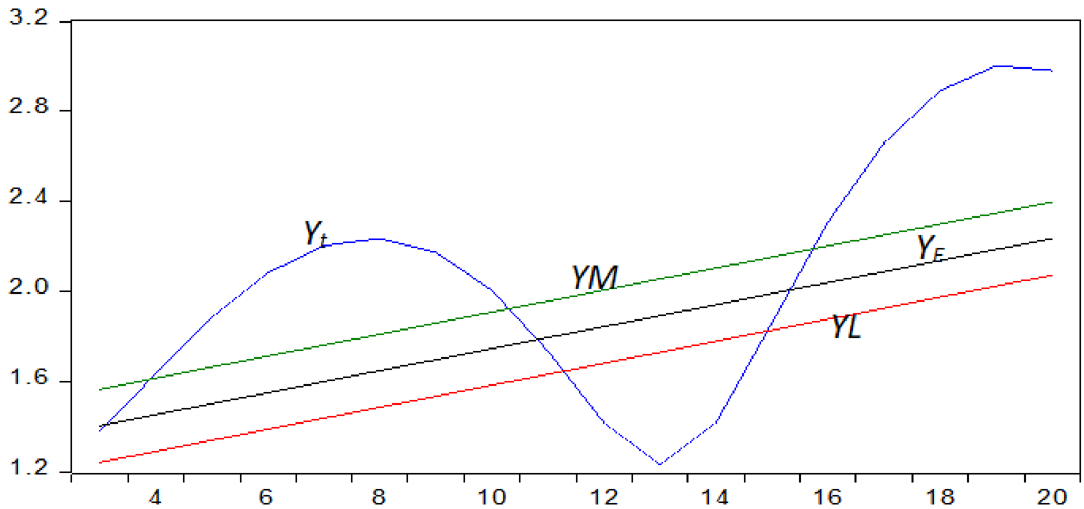
In the Hicks model, the economy is not stationary and exhibits a positive growth rate. As long as $c < 0.6$ and $k < 1$, the fluctuations of the national income Y_t remain inside both limits and are damped as Figure 3 displays.

Figure 3. Damped Fluctuations



For example, if $c = 0.6$ and $k = 0.8$, the fluctuations of income are damped (the national income is running towards its equilibrium value) and remain inside the corridor as long as the marginal propensity to consume is less than 0.6. However, when $c > 0.6$ and $k > 1$, oscillations in national income become explosive.

Figure 4. Explosive Fluctuations



For $c = 0.7$ and $k = 1.2$, explosive fluctuations of income are out of the upper and lower limits.

The Samuelson-Hicks model can exhibit chaos because it implies a second-order difference equation for output. This arises because investment is assumed to depend on the lagged change in output. The key mechanism highlighted by Samuelson is the accelerator effect, which arises because investment depends on the change in output. The assumption that investment depends on the lagged change in output is not essential; the accelerator effect also arises if investment depends on the current change in output. But in that case, chaos does not arise as output is a first-order difference equation, not second-order. Thus, only if the output is a second-order equation, can the occurrence and relevance of chaos be measured by the Lyapunov exponent – which is a useful tool for exhibiting the national income trajectory between the floor and ceiling. Such a trajectory of national income depends on the values of capital coefficient (*k*) and marginal propensity to consume (c).

4. The Lyapunov Exponent: A Useful Tool for Measuring Chaos in the Samuelson-Hicks Model

The Lyapunov exponent can be seen as one of the most relevant tools for showing the occurrence of chaos in dynamical systems, as well as in time series related to economic or financial data. In our paper, we use such a tool for detecting the occurrence of chaos in the Samuelson-Hicks model. Thus, we carry out a Monte-Carlo simulation to analyse national income behaviour which depends on the simulated values of the coefficient for capital, *k*. This method enables us to mathematically determine the aperiodic fluctuations and strange attractors in the Samuelson-Hicks model. However, using the Lyapunov method allows us to describe the trajectory of a macroeconomic variable – but not specifically to reach an economic objective.

The Lyapunov exponent is the quantity that characterises the rate of separation of infinitesimally close trajectories. As mentioned before, it plays an important role in identifying the chaotic degree of the strange attractor (Wu and Baleanu, 2015). The number of Lyapunov exponents equals the number of state variables considered. If we consider a unidimensional system, like in our paper, we may compute one single exponent (Lopez-Jéminez et al., 2002).

A *positive* Lyapunov exponent causes this separation to increase over further iterations and shows a chaotic dynamic. A *negative* Lyapunov exponent indicates an attracting fixed point or periodic cycle, and implies a non-chaotic dynamic characterised by a strange non-chaotic attractor. A Lyapunov exponent that is equal to zero displays sinusoidal oscillations and periodic attractor.

When searching chaos in the Hicks model, we use the Wolf method (1985) to estimate the Lyapunov exponent (called λ_t) using different values for the marginal propensity to consume *c* and the marginal coefficient of capital k.

By this method, we start from an initial condition Y_t in the Hicks model, and we consider a very close value of separation, where the initial distance d_0 is extremely small. The absolute value of d_t after t iteration is:

$$|d_t| = |d_0|e^{\lambda t} \qquad [16]$$

It is equivalent to write:

$$\lambda_t = \lim_{t\to\infty} \frac{1}{t}\left|\frac{\partial d(t)}{\partial do}\right| \qquad [17]$$

We choose the value of the separation $d_0 = 10^{-4}$ and obtain values of λ_t that give the values of the Lyapunov exponent. After a Monte Carlo simulation with 1000 random values of coefficient for capital, *k* (ranging from 0 to 4), we estimate the Average Lyapunov Exponent (*ALE*). Since chaos arises – as output is a second-order difference equation – the marginal propensity to consume (included in the first-order difference equation) is fixed. It takes several values (0.5, 0.6, and 0.8) in Samuelson's original paper (1939, p. 77). We arbitrarily choose $c = 0.8$.

Figure 5. *ALE* evolution with respect to the coefficient for capital

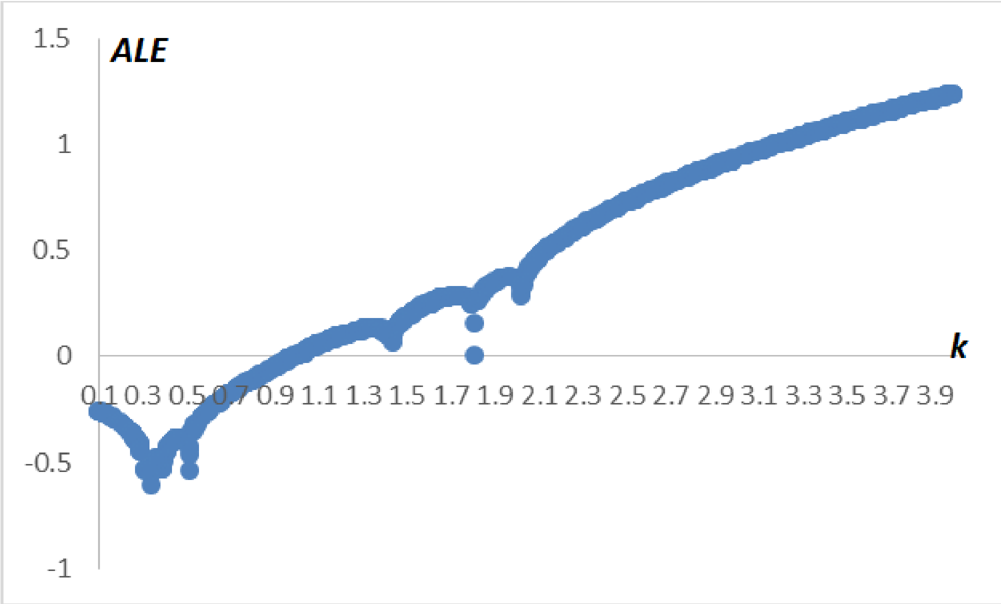


Figure 5 shows that the *ALE* is negative for $0 < k < 1$. This means that the behaviour of the national income (Y_t) exhibits a non-chaotic dynamic characterised by damped oscillations (stationary process with modulus $\rho < 1$). Then, if $k = 1$, the *ALE* is nil meaning that the national income fluctuations are sinusoidal (with the modulus $\rho = 1$). It becomes more relevant when $k > 1$. Thus, values in the region $k > 1$ are much more likely to lead to chaos. However, if $k = 1.8$, *ALE* $= 0$. This means that national income moves from chaos back into order for this particular value. But, in general, from $k > 1.5$ to $k = 4$, national income exhibits an increasingly chaotic dynamic (except for $k = 1.8$). In such a region, the oscillations are explosive and the Lyapunov exponent is strongly positive (with modulus $\rho > 1$).

According to the values of the technical coefficient for capital, which is the key parameter leading the national income to chaos, we can observe the occurrence of several attractors inside or outside both limits.

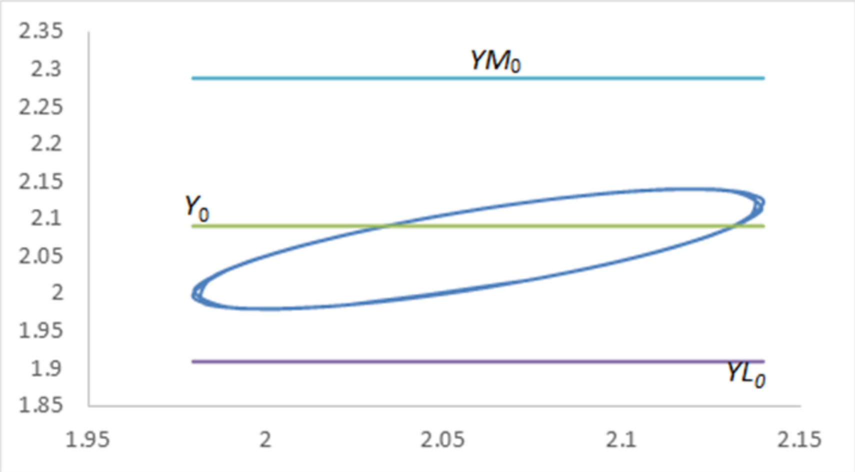
5. Quasi-periodic Attractors in the Hicks Model

Chaos theory involves the concept of the strange attractor for which the trajectories of a variable have a bizarre structure, being neither simple smooth, nor continuous curves but fractals (Puu, 1997). Fractals (Mandelbrot, 1982) could be an indefinite set of unconnected points, or a smooth curve with mathematical discontinuity, or a curve that is fully connected but discontinuous everywhere.

In fact, we have a quasi-periodic attractor when every trajectory winds around endlessly on a torus (Strogatz, 1994).

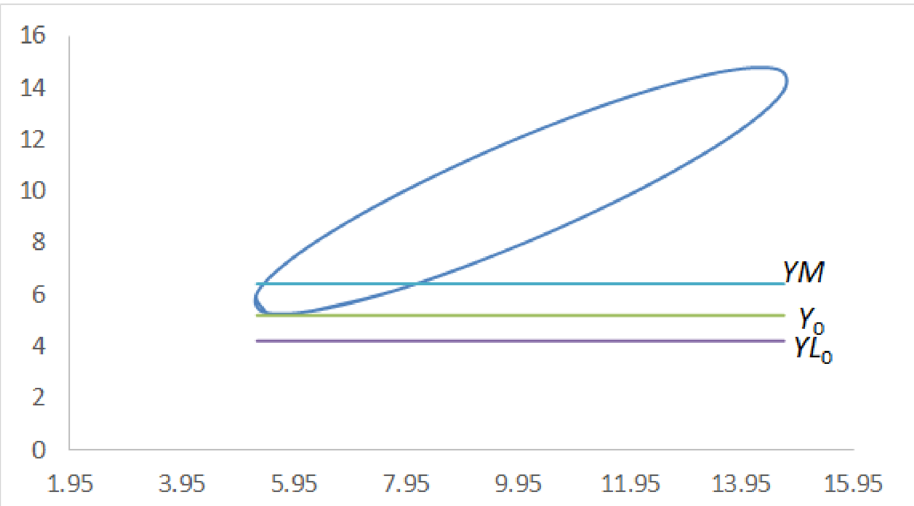
Thus, the following figures represent the strange attractor showing the national income evolution in the space phase where each ordered pair (Y_t, Y_{t-1} ; $t = 2, \ldots, N$) is displayed in the plane (Figures 6a to d). The y-axis represents the values of Y_t and the x-axis, values of Y_{t-1} (Kriz, 2011). The three levels of income (equilibrium income, full employment income, and under-employment income) are represented as well.

Figure 6-a. National Income in the Space Phase: The Perfectly Periodic Attractor between the Two Limits



This Figure shows a perfectly periodic attractor between the upper limit (the income of full employment, called YM_0) and the lower limit (the income of under-employment, called YL_0). Thus, when $c = 0.5$ and $k = 1$, the modulus $\rho = 1$, and oscillations are perfectly sinusoidal. As a result, the periodic attractor is inside both limits. In addition, the Lyapunov exponent is nil meaning that a periodic attractor occurs. However, a rise in the propensity to consume (the coefficient of capital remaining equal to one), pushes the periodic attractor out of the upper limit (Figure 6-b).

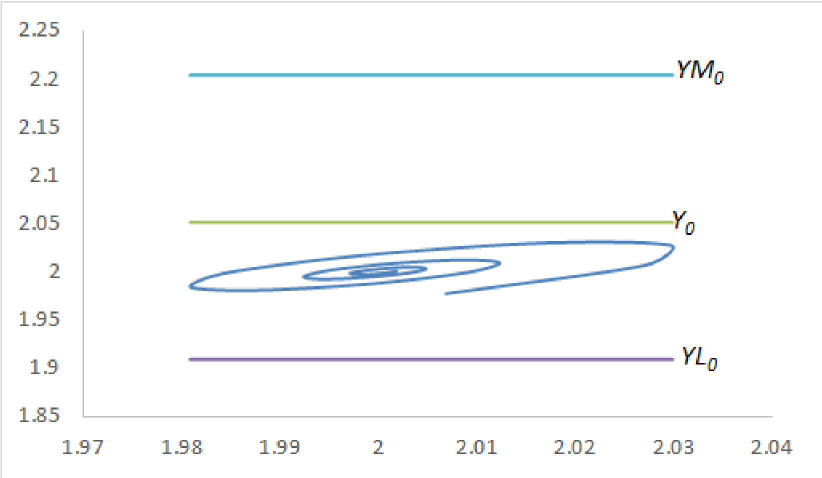
Figure 6-b. National Income in the Space Phase: The Perfectly Periodic Attractor out of the Upper Limit



This Figure exhibits the case where $c = 0.8$ and $k = 1$.

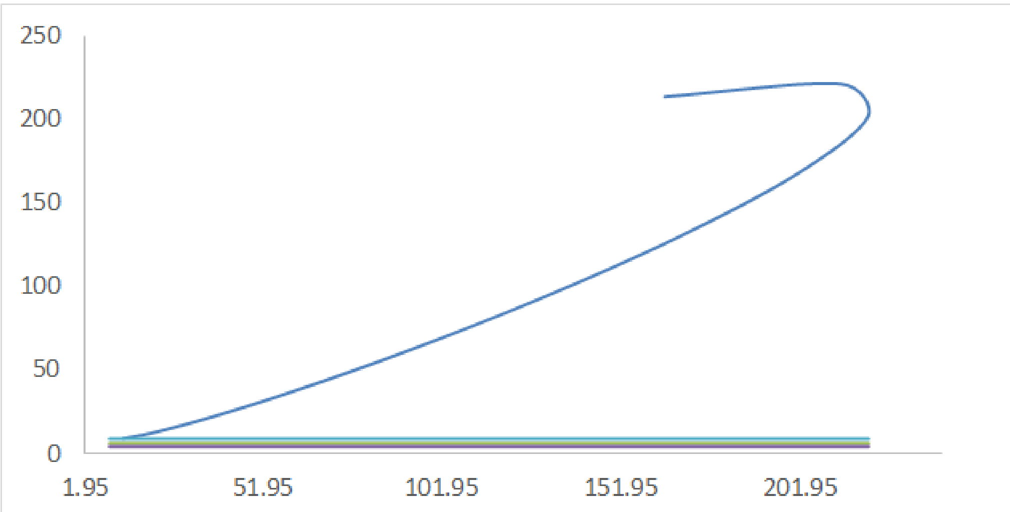
As long as $c \leq 0.5$ and $k = 1$, we have perfectly sinusoidal oscillations and periodic attractor inside both limits. But, if the technical coefficient for capital becomes less than one (with $c \leq 0.5$), the fluctuations are damped (the modulus $p < 1$), and the figure exhibits a strange non-chaotic attractor taking the form of an ellipsoid (Figure 6-c).

Figure 6-c. National Income in the Space Phase: The Occurrence of a Strange Non-Chaotic Attractor



This Figure shows that even though national income exhibits explosive fluctuations in the short run, a strange non-chaotic attractor does exist in the long run that pushes income to regain regular growth. In other words, national income enters the ellipsoid and then remains trapped therein for all future time (Hirsh, Smale and Devaney, 2004). However, when $c > 0.5$ and $k > 1$, the national income records explosive fluctuations and moves away from its trajectory (Figure 6-d).

Figure 6-d. National Income in the Space Phase: It Moves Out of its Trajectory



This figure shows the case where $c = 0.8$ and $k = 1.6$ and displays a chaotic strange attractor that goes beyond both limits. In this hypothesis, national income that starts far from the origin goes away from the ellipsoid and does not return to the equilibrium path. The trajectory of income moves away from the ellipsoid for all future time.

In addition, all figures exhibit a periodic or quasi-periodic attractor (that can be chaotic or not) when national income records oscillations e.g. when the determinant of the polynomial equation Δ is negative. On the contrary, if $\Delta > 0$ (when the parameters $c = 0.8$ and $k > 3$), the evolution of national income becomes explosive without oscillations and the quasi-periodic attractor disappears.

6. Conclusion

Economics can be seen as a complex system 'which evolves towards different attractors depending on the value of its parameters' and 'paves the way to the study of cyclic, non-periodic and chaotic behaviour' (Beker, 2014, p. 221).

In endogenous fluctuations models, such as Samuelson-Hicks, the oscillations of income move between two limits, e.g., the full employment income and under-employment income, depending on the values of the marginal propensity to consume and the technical coefficient for capital. However, the coefficient for capital is the key parameter explaining the relevance of chaos – as output is a second-order difference equation.

Furthermore, according to some values of the Average Lyapunov Exponent (*ALE*), a strange attractor exists and can be chaotic or not.

When the *ALE* is negative, the system has an attracting fixed point or periodic cycle characterised by a strange non-chaotic attractor localised between both limits. When the *ALE* is null, the system displays perfectly sinusoidal fluctuations inside the two limits and presents a perfectly periodic attractor. Chaos and explosive oscillations may occur with certain high values of the two parameters for which the determinant of the polynomial equation remains negative. In such a hypothesis, the *ALE* becomes positive, and the income moves out of equilibrium. Moreover, the attractor becomes chaotic and moves outside both limits. This means that in the Hicks-Samuelson model, the relevance of chaos depends on values taken by the coefficient for capital. For lower values, income oscillations are damped, and the attractor is between the two limits. In addition, the *ALE* is negative, and the strange attractor pushes income to regain regular growth. This illustrates a strange non-chaotic attractor where the income enters the ellipsoid.

The attractor and the oscillations disappear when the determinant of the polynomial equation is positive e.g. when the marginal propensity to consume and the coefficient of capital reach larger values than in the aforementioned case.

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